How to Implement Super-Twisting Controller based on Sliding Mode Observer?

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Outline

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2. STC based on Super-Twisting Output Feedback (STOF)
3. HOSMO based Continuous Control of Perturbed Double Integrator
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Motivation

Consider the dynamical system of the following form

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u + \rho_1 \\
y &= x_1
\end{align*} \]  

(2.1)

where \( y \) is the output variable, \( \rho_1 \) is a non vanishing Lipschitz disturbance and \( |\dot{\rho}_1| < \rho_0 \).

- Our aim is to reconstruct the states of the system and then design super-twisting controller based on the estimated information.
- Although this is already reported in the literature,
- We are going to show that existing methodology is not stand on the sound mathematical background.
Motivation

The super-twisting observer dynamics

\[
\dot{x}_1 = \dot{x}_2 + k_1 |e_1|^\frac{1}{2} \text{sign}(e_1) \\
\dot{x}_2 = u + k_2 \text{sign}(e_1)
\]  

(2.2)

where the error \( e_1 = x_1 - \hat{x}_1 \).

The error dynamics is

\[
\dot{e}_1 = e_2 - k_1 |e_1|^\frac{1}{2} \text{sign}(e_1) \\
\dot{e}_2 = -k_2 \text{sign}(e_1) + \rho_1
\]  

(2.3)

- So \( e_1 \) and \( e_2 \) will converge to zero in finite time \( t > T_0 \), by selecting the appropriate gains \( k_1 \) and \( k_2 \).
- For this, one can say that \( x_1 = \hat{x}_1 \) and \( x_2 = \hat{x}_2 \) after finite time \( t > T_0 \).
Motivation

Consider the sliding manifold of the form

\[ s = c_1 x_1 + \dot{x}_2. \]  \hspace{1cm} (2.4)

The time derivative of (2.4) (for designing the super-twisting control)

\[ \dot{s} = c_1 \dot{x}_1 + \dot{x}_2. \]  \hspace{1cm} (2.5)

After finite time \( t > T_0 \), when observer start extracting the exact information of the states, then one can substitute \( \dot{x}_1 = \dot{x}_2 \).

Also using (2.2) and (2.5), one can further write

\[ \dot{s} = c_1 \dot{x}_2 + u + k_2 \text{sign}(e_1). \]  \hspace{1cm} (2.6)
Motivation

System (5.1) in the co-ordinate of $x_1$ and $s$ by using (2.4) and (2.5)

\[
\dot{x}_1 = s - c_1 x_1 \\
\dot{s} = c_1 \hat{x}_2 + u + k_2 \text{sign}(e_1). \tag{2.7}
\]

Super-twisting control design (which is existing in the literature) as

\[
u = -c_1 \hat{x}_2 - \lambda_1 |s|^{\frac{1}{2}} \text{sign}(s) - \int_0^t \lambda_2 \text{sign}(s) \, d\tau. \tag{2.8}
\]

where $\lambda_1$ and $\lambda_2$ are the designed parameters for the control.

The closed loop system after applying the control (2.8) to (2.7)

\[
\dot{x}_1 = s - c_1 x_1 \\
\dot{s} = -\lambda_1 |s|^{\frac{1}{2}} \text{sign}(s) - \int_0^t \lambda_2 \text{sign}(s) \, d\tau + k_2 \text{sign}(e_1). \tag{2.9}
\]
Motivation

Claim

Second order sliding motion is never start in the (2.9)

Mathematical discussion

- Because of \( \dot{s} \) contains the non-differentiable term \( k_2 \text{sign}(e_1) \).
- Which exclude the possibility of lower two subsystem of (2.9) to act as the super-twisting algorithm.
- So the second order sliding motion (so that \( s = \dot{s} = 0 \) in finite time) cannot be establish.
- In the next, we are going to propose the possible methodology of the control design such that non-differentiable term \( k_2 \text{sign}(e_1) \) is cancel out.
- The lower two subsystem of (2.9) act as the super-twisting and finally second order sliding is achieved.
Proposed method 1

The main aim here, is to design $u$, such that sliding motion occurs in finite time.

**Proposition 1**

The control input $u$ which is defined as

$$u = -c_1 \dot{x}_2 - k_2 \text{sign}(e_1) - \lambda_1 |s|^{\frac{1}{2}} \text{sign}(s) - \int_0^t \lambda_2 \text{sign}(s) d\tau$$  \hspace{1cm} (2.10)

where, $\lambda_1 > 0$ and $\lambda_2 > 0$ are selecting according to (Levant), (Moreno), leads to the establishment second order sliding in finite time, which further implies asymptotic stability of $x_1$ and $x_2$. 
Proposed method 1

Proof

The closed loop system after substituting (2.10) into (2.7)

\[
\dot{x}_1 = s - c_1 x_1 \\
\dot{s} = -\lambda_1 |s|^{\frac{1}{2}} \text{sign}(s) + \nu \\
\dot{\nu} = -\lambda_2 \text{sign}(s)
\]

(2.11)

Last two equation of (2.11) has same structure as super-twisting. Therefore, one can easily observe that after finite time \( t > T_0 \), \( s = \dot{s} = 0 \).

The closed loop system is given as

\[
\dot{x}_1 = -c_1 x_1 \\
\dot{\hat{x}}_2 = -c_1 x_1
\]

(2.12)

Therefore, both states \( x_1 \) and \( \hat{x}_2 \) are asymptotic stability by choosing \( c_1 > 0 \). Also, when observer estimating the exact state \( \hat{x}_2 = x_2 \) after finite time, then \( x_2 \) also going to zero simultaneously as \( \hat{x}_2 \).
Existing result: STC based on Super-Twisting Output Feedback (STOF)

Step-1
Consider the following sliding sliding surface

\[ s = c_1 x_1 + x_2 \]  \hspace{1cm} (3.1)

assuming that all states information are available.

Step-2
To realizing the control expression based on super-twisting, take the first time derivative of sliding surface \( s \) using (3.1)

\[ \dot{s} = c_1 \dot{x}_1 + \dot{x}_2 \]  \hspace{1cm} (3.2)

Step-3
Now substitute \( \dot{x}_1 \) and \( \dot{x}_2 \) from (5.1) into (3.2),

\[ \dot{s} = c_1 x_2 + u + \rho_1 \]  \hspace{1cm} (3.3)
Step-4

Now design control as

\[ u = -c_1 x_2 - \lambda_1 |s|^\frac{1}{2} \text{sign}(s) - \int_0^t \lambda_2 \text{sign}(s) d\tau \]  \hspace{1cm} (3.4)

After substituting the control (3.4) into (3.3),

\[ \dot{s} = -\lambda_1 |s|^\frac{1}{2} \text{sign}(s) + \nu \]
\[ \dot{\nu} = -\lambda_2 \text{sign}(s) + \dot{\rho}_1. \]  \hspace{1cm} (3.5)

Now select \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \) according to (moreno2012), which leads to second order sliding in finite time provided \( \rho_1 \) is Lipschitz and \( |\dot{\rho}_1| < \rho_0 \).

When \( s = 0 \), then \( x_1 = x_2 = 0 \) asymptotically same as discussed above by selecting \( c_1 > 0 \).
Existing result: STC based on Super-Twisting Output Feedback (STOF)

- The control (3.4) is not implementable because we do not have information of $x_2$, so replace $x_2$ by $\hat{x}_2$.

- It is argued that after finite time $x_1 = \hat{x}_1$ and $x_2 = \hat{x}_2$, therefore control signal applied to original system (5.1) is

$$u = -c_1 \hat{x}_2 - \lambda_1 |\hat{s}|^{\frac{1}{2}} \text{sign}(\hat{s}) - \int_0^t \lambda_2 \text{sign}(\hat{s}) d\tau$$

(3.6)

where $\hat{s} = c_1 \hat{x}_1 + \hat{x}_2$,

- Without considering the the dynamics of $\dot{\hat{x}}_2$ for which control derivation is explicitly dependent and it contains the discontinuous term $k_2 \text{sign}(e_1)$.

- One can easily see that average value of this discontinuous term is equal to negative of the disturbance.

- So control (3.6) we are applying for the real system is only approximate not the exact. However, the exact controller is always discontinuous which already discussed and mathematically proved in the above section.
Higher Order Sliding Mode Observer based Continuous Control of Perturbed Double Integrator

This method gives the correct way to implement continuous STC, when only output information of the perturbed double integrator (5.1) is available.

The HOSMO dynamics to estimate the states for the system (5.1) is given as

\[\begin{align*}
\dot{x}_1 &= \dot{x}_2 + k_1 |e_1|^{\frac{2}{3}} \text{sign}(e_1) \\
\dot{x}_2 &= \dot{x}_3 + u + k_2 |e_1|^{\frac{1}{3}} \text{sign}(e_1) \\
\dot{x}_3 &= k_3 \text{sign}(e_1)
\end{align*}\] (4.1)

Let us define the error \(e_1 = x_1 - \hat{x}_1\) and \(e_2 = x_2 - \hat{x}_2\) and the error dynamics is

\[\begin{align*}
\dot{e}_1 &= e_2 - k_1 |e_1|^{\frac{2}{3}} \text{sign}(e_1) \\
\dot{e}_2 &= -\dot{x}_3 - k_2 |e_1|^{\frac{1}{3}} \text{sign}(e_1) + \rho_1 \\
\dot{x}_3 &= k_3 \text{sign}(e_1)
\end{align*}\] (4.2)
Now define the new variable $e_3 = -\hat{x}_3 + \rho_1$, if $\rho_1$ is Lipschitz and $|\dot{\rho}_1| < \rho_0$.

One can further write (4.2) as

$$
\begin{align*}
\dot{e}_1 &= e_2 - k_1|e_1|^{\frac{2}{3}} \text{sign}(e_1) \\
\dot{e}_2 &= e_3 - k_2|e_1|^{\frac{1}{3}} \text{sign}(e_1) \\
\dot{e}_3 &= -k_3 \text{sign}(e_1) + \dot{\rho}_1
\end{align*}
$$

(4.3)

- So $e_1$, $e_2$ and $e_3$ will converge to zero in finite time $t > T_0$, by selecting the appropriate gains $k_1$, $k_2$ and $k_3$ (Levant).
- After the convergence of error, one can find that $x_1 = \hat{x}_1$, $x_2 = \hat{x}_2$ and $\hat{x}_3 = \rho_1$ after finite time $t > T_0$. 

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Consider the sliding surface (2.4) and its time derivative is

\[ \dot{s} = c_1 \dot{x}_1 + \dot{x}_2. \] (4.4)

After finite time \( t > T_0 \), when observer start extracting the exact information of the states, then one can substitute \( \dot{x}_1 = \hat{x}_2 \).

Also using (4.2) and (4.4), one can further write

\[ \dot{s} = c_1 \hat{x}_2 + u + k_2 |e_1|^{\frac{1}{3}} \text{sign}(e_1) + \int_0^t k_3 \text{sign}(e_1) d\tau \] (4.5)

The system (5.1) in the co-ordinate of \( x_1 \) and \( s \) by using (2.4) and (4.5)

\[ \dot{x}_1 = s - c_1 x_1 \]

\[ \dot{s} = c_1 \hat{x}_2 + u + k_2 |e_1|^{\frac{1}{3}} \text{sign}(e_1) + \int_0^t k_3 \text{sign}(e_1) d\tau \] (4.6)
HOSMO based Proposed method

**Proposition 2**

The control input $u$ which is defined as

$$u = -c_1 \dot{x}_2 - k_2 |e_1|^3 \text{sign}(e_1) - \int_0^t k_3 \text{sign}(e_1) \, d\tau - \lambda_1 |s|^\frac{1}{2} \text{sign}(s)$$

$$- \int_0^t \lambda_2 \text{sign}(s) \, d\tau$$

or

$$u = -c_1 \dot{x}_2 - \int_0^t k_3 \text{sign}(e_1) \, d\tau - \lambda_1 |s|^\frac{1}{2} \text{sign}(s) - \int_0^t \lambda_2 \text{sign}(s) \, d\tau$$

(4.7)

(4.8)

Because observer is much faster, which makes $e_1 = 0$ in finite time.

- If $\lambda_1 > 0$ and $\lambda_2 > 0$ are selecting according to (Levant), (Moreno), leads to the establishment of $s$ equal to zero in finite time, it further implies asymptotic stability of $x_1$ and $x_2$.

Proof is the same as the Proposition 1.
Discussion of HOSMO based STC Design

- It is clear from the STC control (4.7) expression based on HOSMO (4.1) is continuous.

- Also, when we design STC control based on HOSMO then one has to tune only observer gain according to the first derivative of disturbance, because it is necessary for the convergence of the error variables of the HOSMO.

- However, during controller design there is no explicit gain condition for the $\lambda_2$ with respect to disturbances.

- One can also observe that STC (3.6) design based STOF (2.2), (which is propagating in the literature without any sound mathematical justification) requires two gains.

- One is the STO observer gain $k_2$ based on the explicit maximum bound of the direct disturbance and another is $\lambda_2$, for the STC based on the maximum bound of the derivative of disturbance.
Discussion of HOSMO based STC Design

Some observation

- From the above observation that sound mathematical analysis reduces the **two gains conditions with respect to disturbance** by simply **one gain condition**.

- Also the **precision of the sliding manifold is much improved** by using the HOSMO based STC rather than STO based STC.

- Due to the **increase of this precision of sliding variable precision of the states are also much effected**.

- In other word if we talk about stabilization problem, **then states are much closer to origin in the case of HOSMO based STC** rather than STO based STC.

- We only talk about closeness of states variable with respect to equilibrium point, because only asymptotic stability is possible in the both of design methodology.
Numerical Simulation

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = u + \rho_1 \]
\[ y = x_1 \]  

(5.1)

For the simulation, initial conditions of perturbed double integrator, for the all three cases, STC-STO, STC-STOF and STO-HOSMO, is taken as \( x_1 = 10, \) \( x_2 = 0 \) and \( \rho_1 = 2 + 3 \sin(t) \). Other gains for the all three cases are selected as follows

- **STC-STO**
  - STC gains \( k_1 = 3 \) and \( k_2 = 4 \)
  - STO gains \( \lambda_1 = 3.5 \) and \( \lambda_2 = 6 \)

- **STC-STOF**
  - STC gains \( k_1 = 2 \) and \( k_2 = 1 \)
  - STO gains \( \lambda_1 = 3.5 \) and \( \lambda_2 = 6 \)

- **STC-HOSMO**
  - STC gains \( k_1 = 2 \) and \( k_2 = 1 \)
  - STO gains \( \lambda_1 = 6, \lambda_2 = 11 \) and \( \lambda_3 = 6 \)
Numerical Simulation: without noise

**Figure**: Evolution of output w.r.t. time for the STC based on HOSMO, STOF and STO
Numerical Simulation: without noise

Figure: Evolution of sliding manifold w.r.t. time for the STC based on HOSMO, STOF and STO
Numerical Simulation: without noise

Figure: Evolution of error w.r.t. time using STO and HOSMO
Numerical Simulation: without noise

Figure: Evolution of control STC based on HOSMO and STOF w.r.t. time
Numerical Simulation: without noise

Figure: Evolution of control STC based on STO w.r.t. time
Numerical Simulation: with noise

Figure: Evolution of output w.r.t. time for the STC based on HOSMO, STOF and STO under noisy measurement
Numerical Simulation: with noise

**Figure**: Evolution of sliding manifold w.r.t. time for the STC based on HOSMO, STOF and STO under noisy measurement.
Numerical Simulation: with noise

Figure: Evolution of error w.r.t. time for the STC based on HOSMO, STOF and STO under noisy measurement
Numerical Simulation: with noise

Figure: Evolution of control STC based on HOSMO and STOF w.r.t. time under noisy measurement
Numerical Simulation: with noise

Figure: Evolution of control STC based on STO w.r.t. time under noisy measurement
Conclusion

- It is shown in the paper that, if one wants to implement absolutely continuous STC signal for the perturbed double integrator, the derivative of the chosen manifold must be Lipschitz in the time.
- Therefore, we have the need of second order observer/differentiators in this case.
- The same is also true for the higher order perturbed chain of integrators, when we want to synthesize absolutely continuous STC signal under the output information.
- Numerical simulations are also presented to support the effectiveness of the proposed methodology.
Thank You!