Practical Relative Degree in SMC systems: Frequency Domain Approach

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   - Problem Statement

2 Practical Relative Degree

3 Performance Margins
   - Definition of Performance Margins
   - Performance margins for conventional SMC
   - Performance margins for 2-SM (Twisting Algorithm)

4 Practical Relative Degree with Performance Margins Requirement

5 Conclusions
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Introduction

SMC

Model Unknown

Relative Degree Unknown

Chattering

Tolerance Limits

Frequency Domain Analysis

Unmodeled Dynamics (Fractal)

Supposition “Sliding variable converge to a Limit Cycle”

SMC

2-SMC

Tolerance Limits

Practical Relative Degree (k)

Tolerance Limits + Performance Margins

k-SMC (k+1)-SMC

PRD with Performance Margins Requirement

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2. Boiko, et al. TAC 2005
3. Hernandez, CEP 2013
4. Boiko, JFI 2013
5. Shtessel, et al. CDC 96
Problem Statement

Control System

A1. Plant has unknown relative degree $r$.
A2. $W(j\omega)$ of the linear plant may be obtained.
A3. Linear plant has low pass filter properties
A4. Amplitude and phase frequency characteristics of $W(j\omega)$ are monotonously decreasing functions, i.e. $|W(j\omega_1)| > |W(j\omega_2)|$ and $\text{arg } W(j\omega_1) > \text{arg } W(j\omega_2)$ for $\omega_1 < \omega_2$
A5. Describing Function of SMC may be obtained and it depends only on the amplitude $A$
Problem Statement

Effects of Unmodeled Dynamics

Presence of unmodeled dynamics in the system increase the relative degree.

- The output $y$ will not converge to zero but to a limit cycle [Shtessel, et al. 96], [Boiko, et al. 05]
- HB equation gives a solution and limit cycles in systems controlled by SMC may be predicted.

Tolerance Limits

The frequency $\omega_c$ and amplitude $A_c$ are the tolerance limits of the acceptable limit cycle of the output $y$, so that self-sustained oscillations of the output $y$ with the amplitudes $A \leq A_c$ and the frequencies $\omega \geq \omega_c$ yield the acceptable performance of the closed loop system in the real sliding mode [Utkin 09]
**Problem Statement**

**Motivation Example**

Suppose system has $r > 2$ and it is controlled by SMC and 2-SMC

Case 3. Which controller is to be selected for the implementation?

<table>
<thead>
<tr>
<th>Case</th>
<th>Controller</th>
<th>SMC</th>
<th>2-SMC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>$A &gt; A_c$ and/or $\omega &lt; \omega_c$</td>
<td>$A &gt; A_c$ and/or $\omega &lt; \omega_c$</td>
<td></td>
</tr>
<tr>
<td>Case 2</td>
<td>$A &gt; A_c$ and/or $\omega &lt; \omega_c$</td>
<td>$A \leq A_c, \omega \geq \omega_c$</td>
<td></td>
</tr>
<tr>
<td>Case 3</td>
<td>$A \leq A_c, \omega \geq \omega_c$</td>
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5 Conclusions
Practical Relative Degree (PRD) is understood as the smallest order $r$ of SMC, that yields a predicted limit cycle in the closed-loop system with the amplitude $A \leq A_c$, $A_c > 0$ and the frequency $\omega \geq \omega_c$, $0 < \omega_c < \infty$. 
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Definition of Performance Margins

**Performance Phase Margin**

The Performance Phase Margin (PPM) in the system is the maximal additional phase shift in $W(j\omega)$ that the closed loop system can tolerate for its output $y$ to exhibit the acceptable limit cycle with $A \leq A_c$, $A_c > 0$ and $\omega \geq \omega_c$, $0 < \omega_c < \infty$ in the real sliding mode.

**Performance Gain Margin**

The Performance Gain Margin (PGM) in the system is the maximum additional gain in $W(j\omega)$ that the closed loop system can tolerate for its output $y$ to exhibit the acceptable limit cycle with $A \leq A_c$, $A_c > 0$ and/or $\omega \geq \omega_c$ $0 < \omega_c < \infty$ in the real sliding mode.
DF conventional SMC

Let the controller be

\[ u = -\alpha \text{sign}(y) \]  

(1)

DF is \( N(A) = 4\alpha / \pi A \), where \( A \) is the amplitude of \( y \).

Harmonic Balance equation is

\[ \text{Re} \{ W(j\omega) \} + j \text{Im} \{ W(j\omega) \} = -\frac{\pi A}{4\alpha} \]  

(2)

where \( \omega \) is the frequency of the output of system.
Performance Phase Margin Conventional SMC

PPM SMC

Performance Phase Margin Method

1. Locate maximum amplitude $A_c$ and the minimum frequency $\omega_c$ in $-1/N(A)$ and $W(j\omega)$, respectively.

3. Plot a circle with radius $A_c$.

3(a) If the frequency $\omega_c$ is located outside of the circle, the PPM should be obtained as the angle formed between the intersection of the circle with $W(j\omega)$ and the negative real axis.

3(b) If the frequency $\omega_c$ is located inside of the circle, the PPM should be obtained as the angle formed between the vector associated to $\omega_c$ and the negative real axis.
Consider the gain $K > 1$ in the harmonic balance equation

$$K \cdot W(j\omega) = -\frac{1}{N(A)}$$

Solve the harmonic balance equation for $K$ with $A_c$ and $\omega_c$ known,

$$K \Re(W(j\omega)) = -\frac{\pi A}{4\alpha}$$

$$K \Im(W(j\omega)) = 0$$

Value of $K \geq 1$ which satisfy HB equation system with $r \leq 2$ is $K \rightarrow \infty$

$\text{PGM} \rightarrow \infty$. 
1. Consider the gain $K > 0$ in the harmonic balance equation

$$K \cdot W(j\omega) = -\frac{1}{N(A)}$$

2. Solve the harmonic balance equation for $K$ with $A_c$ and $\omega_c$ known,

$$K = -\frac{1}{N(A_c)} \frac{1}{W(j\omega_c)}$$

3. The value of $K$ is the PGM of the system for the acceptable amplitude $A_c$. 
Let the controller be

\[ u = -c_1 \text{sign}(y) - c_2 \text{sign}(\dot{y}) \]  

(3)

where \( c_1 > c_2 > 0 \).

DF is

\[ N(A) = \left(\frac{4}{\pi A}\right)(c_1 + jc_2), \]

where \( A \) is the amplitude of the output \( y \).

Harmonic Balance equation is

\[ W(j\omega) = -\frac{A\pi}{4} \frac{c_1 - jc_2}{c_1^2 + c_2^2} \]  

(4)

where \( \omega \) is the frequency of the output \( y \).
Performance Margins for 2-SM

Figure: For Twisting Algorithm. a) PPM, b) PGM
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Suppose that \( PPM_c \) and \( PGM_c \) are the acceptable performance phase and gain margins. Practical Relative Degree

Practical Relative Degree (PRD) is understood as the smallest order \( r \) of SMC, that yields a predicted limit cycle in the closed-loop system with the amplitude \( A \leq A_c, A_c > 0 \) and the frequency \( \omega \geq \omega_c, 0 < \omega_c < \infty \) while the \( PPM \geq PPM_c \) and \( PGM \geq PGM_c \), where \( PPM_c \) and \( PGM_c \) are the acceptable performance phase and gain margins.

Therefore, the controller, which order corresponds to the calculated PRD, is to be implemented.
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Practical relative degree in LTI SISO systems controlled by SMC is defined based on tolerance limits in terms of amplitude and frequency of a possible limit cycle on sliding variable.

PRD is understood as the smallest order of SMC that yields acceptable performance.

The proposed identification of practical relative degree can be used for the SMC design for systems treated as a black box.

The notion is given in terms of performance margins which can be useful when the PRD could be affected by parameter changes or errors in the frequency response identification.
Conclusions

SMC

Model

Unknown

Relative

Degree

Unknown

Unmodeled

Dynamics

(Fractal)

Chattering

Tolerance

Limits

Frequency

Domain

Analysis

Supposition

“Sliding

variable

converge to

a

Limit Cycle”

SMC

2-SMC

k-SMC

(k+1)-SMC

Tolerance

Limits

Tolerance

Limits

PRD with

Performance

Margins

Requirement

Frequency

Domain

Analysis

Boiko, et al. TAC 2004
Boiko, et al. TAC 2005
Hernandez, CEP 2013
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Shtessel, et al. CDC 96

Practical

Relative

Degree

(k)

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Tolerance

Limits

+ Performance

Margins

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