Higher Order Super-Twisting Algorithm

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Outline

Motivation

Consider the second order system

$$\ddot{\sigma} = u + \rho_1 \tag{2.1}$$

where ρ_1 is a non vanishing Lipschitz disturbance and $|\dot{\rho}_1| < \rho_0$.

Algorithm	Control Signal	Information	Stability	Chattering
First SMC	Discontinuous	σ and $\dot{\sigma}$	Asymptotic	Yes
STC	Continuous	σ and $\dot{\sigma}$	Asymptotic	No
Twisting	Discontinuous	σ and $\dot{\sigma}$	Finite time	Yes
Third SMC	Continuous	σ and $\dot{\sigma}$ and disturbance	Finite time	No

Table : Different control strategies for the second order uncertain integrator with output σ and its derivative $\dot{\sigma}$

- It is clear from the table that finite time control under the absolutely continuous control signal without explicit knowledge of disturbance is still unexplored.
- Similar kind of situation is also true for the system with higher relative degree.

Generalized Order Super-Twisting

Generalized order Super-twisting which has following properties:

- finite time convergence for the set $\sigma, \dot{\sigma}, ..., \sigma^{(r)}$ where σ represents the output and r is the relative degree of the system with respect to output using information of $\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}$ which generates the absolutely continuous control signal for the arbitrary relative degree;
- compensates theoretically exactly Lipschitz in time on the system trajectories uncertainties/perturbations;
- precision of the output σ corresponding to $(r+1)^{th}$ order sliding mode;

Notation

In this paper the following notation is used, for a real variable $z \in \mathbb{R}$ to a real power $p \in \mathbb{R}$, $\lfloor z \rceil^p = |z|^p sgn(z)$, therefor $\lfloor z \rceil^2 = |z|^2 sgn(z) \neq z^2$. If p is an odd number, this notation does not change the meaning of the equation, i.e. $\lfloor z \rceil^p = z^p$. Therefore

$$\lfloor z \rceil^0 = sgn(z), \quad \lfloor z \rceil^0 z^p = |z|^p, \quad \lfloor z \rceil^0 |z|^p = \lfloor z \rceil^p$$
$$\lfloor z \rceil^p \lfloor z \rceil^q = |z|^p sgn(z)|z|^q sgn(z) = |z|^{p+q} \qquad (2.2)$$

Also, $\sigma = x_1$ represents the output for the generalized n-STA.

Definition

Following standard definition existing in literature [**?**]:

Definition

A vector field $f : \mathrm{R}^n \to \mathrm{R}^n$ (or a differential inclusion) is called homogeneous of degree $\delta \in \mathrm{R}$ with the dilatation $d_\kappa : (x_1, x_2, \cdots, x_n) \mapsto (\kappa^{\varrho_1} x_1, \kappa^{\varrho_2} x_2, \cdots, \kappa^{\varrho_n} x_n)$, where $\varrho = (\varrho_1, \varrho_2, \cdots, \varrho_n)$ are some positive numbers (called the weights), if for any $\kappa > 0$ the following identity $f(x) = \kappa^{-\delta} d_\kappa^{-1} f(d_\kappa x)$ holds.

Definition

A scalar function $V : \mathrm{R}^n \to \mathrm{R}$ is called homogeneous of degree $\delta \in \mathrm{R}$ with the dilatation d_κ if for any $\kappa > 0$ the following identity $V(x) = \kappa^{-\delta} V(d_\kappa x)$ holds.

Higher Order STA

In this section generalization of STA is presented.

- For the simplicity of notation algorithm is expressed in the term of $x_1, x_2, \cdots, x_n$
- where $\sigma = x_1$ is the output.

2-STA is given as follows

$$\begin{aligned} \dot{x}_1 &= -k_1 |x_1|^{\frac{1}{2}} \text{sign}(x_1) + x_2 \\ \dot{x}_2 &= -k_2 \text{sign}(x_1) + \rho \end{aligned} \tag{3.1}$$

where x_1, x_2 represent the states and the perturbation ρ satisfied $|\rho| \leq \Delta$.

Higher Order STA

3-STA is proposed as follows

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 \left|\phi_1\right|^{1/2} \operatorname{sign}\left(\phi_1\right) + x_3 \\ \dot{x}_3 &= -k_3 \operatorname{sign}\left(\phi_1\right) + \rho \end{aligned} \tag{3.2}$$

where $\phi_1 = x_2 + k_2 |x_1|^{2/3} \operatorname{sign}(x_1)$,
x_1, x_2, x_3 represent the states and the perturbation ρ satisfied $|\rho| \leq \Delta$.

Higher Order STA

4-STA is proposed as follows

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -k_1 \left|\phi_2\right|^{1/2} \operatorname{sign}\left(\phi_2\right) + x_4 \\
\dot{x}_4 &= -k_4 \operatorname{sign}\left(\phi_2\right) + \rho
\end{aligned} \tag{3.3}
$$

where

$$
\phi_2 = x_3 + k_3 \left(|x_1|^3 + |x_2|^4\right)^{\frac{1}{6}} \operatorname{sign}\left(x_2 + k_2 |x_1|^{\frac{3}{4}} \operatorname{sign}(x_1)\right) \tag{3.4}
$$

and x_1, x_2, x_3, x_4 represent the states and the perturbation ρ satisfied $|\rho| \leq \Delta$.

Higher Order STA

5-STA is proposed as follows

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -k_1 \left|\phi_3\right|^{1/2} \operatorname{sign}\left(\phi_3\right) + x_5 \\
\dot{x}_5 &= -k_5 \operatorname{sign}\left(\phi_3\right) + \rho
\end{aligned} \tag{3.5}$$

where

$$\phi_3 = x_4 + k_4 \left[\left(|x_1|^{12} + |x_2|^{15} + |x_3|^{20} \right)^{\frac{1}{30}} \operatorname{sign}\left(l_1\right) \right]$$

and

$$l_1 = x_3 + k_3 \left(|x_1|^{12} + |x_2|^{15} \right)^{\frac{1}{20}} \operatorname{sign}\left(x_2 + k_2 |x_1|^{\frac{4}{5}} \operatorname{sign}(x_1) \right)$$

and x_1, x_2, x_3, x_4, x_5 represent the states and the perturbation ρ satisfied $|\rho| \leq \Delta$.

Higher Order STA

n-STA is proposed as follows

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{n-1} &= -k_1 \left|\phi_{n-2}\right|^{1/2} \operatorname{sign}\left(\phi_{n-2}\right) + x_n \\
\dot{x}_n &= -k_n \operatorname{sign}\left(\phi_{n-2}\right) + \rho
\end{aligned} \tag{3.6}$$

where ϕ_{n-2} we define later part of the paper, $x_1, x_2, \cdots, x_n$ represent the states and the perturbation ρ satisfied $|\rho| \leq \Delta$.

Higher Order STA

Definition of ϕ_{n-2} is given as follows:-

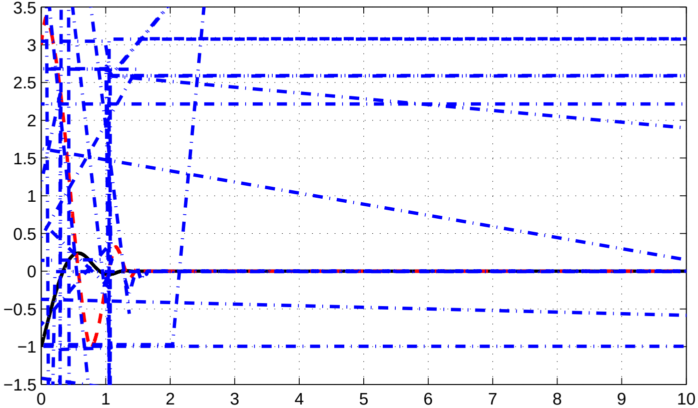
- $$R_{1,r-1} = |x_1|^{\frac{r}{r+1}}$$
  where r represents the relative degree of algorithm with respect to x_1.
- $$R_{i,r-1} = \left| |x_1|^{r_1} + |x_2|^{r_2} + \cdots + |x_{i-2}|^{r_{i-2}} \right|^{q_i}$$
  where $i = 2, 3, \cdots, (r-1)$, $r_1, r_2, \cdots, r_{i-2}$ and q_i is designed parameter based on the homogeneity weight of the x_{i+1}.
- $$S_{0,r-1} = x_1$$
  $$S_{1,r-1} = x_2 + k_2 R_{1,r-1}\text{sign}(x_1)$$
  $$S_{i,r-1} = x_{i+1} + k_{i+1} R_{i,r-1}\text{sign}(S_{i-1,r-1})$$
  where $i = 2, 3, \cdots, (r-1)$
- Finally $\phi_{n-2} = s_{r-1,r-1}$.

Simulation of 3-STA and 4-STA

Under the following value of initial conditions and gains

- 3-STA
  - initial conditions $x_1(0) = -1, x_2(0) = -3$ and $x_3(0) = 1$
  - gains $k_1 = 6, k_2 = 4$ and $k_3 = 4$
- 4-STA
  - initial conditions $x_1(0) = -1, x_2(0) = 3, x_3(0) = 1$ and $x_4(0) = 1$
  - gains $k_1 = 4, k_2 = 2, k_3 = 1$ and $k_4 = 2$

Simulation of 3-STA and 4-STA



Simulation of 3-STA and 4-STA

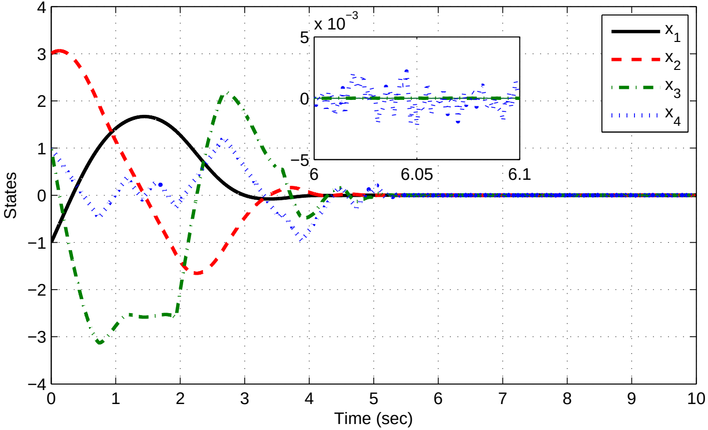


Figure : Evolution of States of 4-STA w.r.t. time

Discussion about 3-STA and other Generalized STA

- 3-STA (3.2) is homogeneous of degree $\delta_f = -1$ with weights $\varrho = [3, 2, 1]$, and its solution can be provide in the sense of Flippov.
- The main advantage of this algorithm is that, output (x_1) and its derivative (x_2) information are only needed for the finite time convergence of all three variables x_1, x_2 and x_3.
- Proposed algorithm can be work as controller for the uncertain system with relative degree 2 with respect to, output in the case of 3-STA.
- Similarly, n-STA homogeneous of degree $\delta_f = -1$ with weights $\varrho = [n, n-1, \cdots, 2, 1]$ and used for the uncertain system with relative degree $n-1$ with respect to output.
- The main idea behind construction of this algorithm is that add one extra discontinuous integral term which is able to reconstruct the perturbation and also nullify.
- But it is necessary that perturbations must be Lipschitz continuous, meaning is that the first derivative is exit almost everywhere and its also be bounded, but perturbations might not be bounded.
- It is necessary to specify here that large number of second order uncertain systems contain this class of perturbations.

Convergence condition of 3-STA

Consider the following continuous candidate Lyapunov function for the stability analysis of (3.2)

$$
\begin{aligned}
V(x) = {} & p_1 |x_1|^{\frac{4}{3}} - p_{12} \lfloor x_1 \rceil^{\frac{2}{3}} \left(x_2 + k_2 \lfloor x_1 \rceil^{2/3} \right) \\
& + p_2 \left| x_2 + k_2 \lfloor x_1 \rceil^{2/3} \right|^2 + p_{13} \lfloor x_1 \rceil^{\frac{2}{3}} \lfloor x_3 \rceil^2 \\
& - p_{23} \left(x_2 + k_2 \lfloor x_1 \rceil^{2/3} \right) \lfloor x_3 \rceil^2 + p_3 |x_3|^4
\end{aligned} \tag{4.1}
$$

- $V(x)$ is homogeneous of degree $\delta_V = 4$, with weights $\varrho = [3, 2, 1]$.
- It is differentiable everywhere but it is not locally Lipschitz at $x_1 = 0$.

Convergence condition of 3-STA

- Our main aim to derive the conditions for the coefficient $(p_1, p_{12}, p_2, p_{13}, p_{23}, p_3)$ and for the gains (k_1, k_2, k_3) of the third order super-twisting algorithm (3.2).
- Such that $V(x) > 0$ and time derivative of Lyapunov function (4.1) along (3.2) is negative definite ($\dot{V} < 0$ for all $x \in \mathbb{R}^3, x \neq 0$).

Lyapunov function (4.1) can also be expressed as in quadratic form in the vector $\Xi^T = \left[\lfloor x_1 \rceil^{\frac{2}{3}} \ \ \phi \ \ \lfloor x_3 \rceil^2 \right]$, where $\phi = \left(x_2 + k_2 \lfloor x_1 \rceil^{2/3} \right)$, i.e.

$$V(x) = \Xi^T P \Xi, \quad \text{where} \ \ P = \begin{bmatrix} p_1 & -\frac{1}{2}p_{12} & \frac{1}{2}p_{13} \\ -\frac{1}{2}p_{12} & p_2 & -\frac{1}{2}p_{23} \\ \frac{1}{2}p_{13} & -\frac{1}{2}p_{23} & p_3 \end{bmatrix} \tag{4.2}$$

Proposition-1

Consider the continuous and homogeneous function $V(x)$ given by (4.2). $V(x)$ is positive definite and radially unbounded if and only if ($P > 0$)

$$p_1 > 0, \quad p_1 p_2 > \frac{1}{4} p_{12}^2,$$
$$p_1 \left(p_2 p_3 - p_{23}^2 \right) + \frac{p_{12}}{2} \left(-\frac{p_{12} p_3}{2} + \frac{p_{13} p_{23}}{4} \right) + \frac{p_{13}}{2} \left(\frac{p_{12} p_{23}}{4} - \frac{p_2 p_{13}}{2} \right) > 0. \tag{4.3}$$

In this case $\dot{V}(x)$ satisfies the differential inequalities

$$\dot{V} \leq -\kappa V^{3/4} \tag{4.4}$$

for some positive κ and it is a Lyapunov function for the system (3.2), whose trajectories converges in finite time to the origin $x = 0$ for every value of the perturbation $|\rho| < \Delta$. The convergence time of a trajectory starting at the initial condition x_0 can be estimated as

$$T(x_0) \leq \frac{4}{\kappa} V^{\frac{1}{4}}(x_0) \tag{4.5}$$

Proof

Proposition Proof

It is obvious that by taking (4.1) as Lyapunov candidate function and calculating first time derivative along (3.2), one can always find κ for the set of gains k_1, k_2, k_3 for which states of 3-STA (3.2) converges to the equilibrium point in the finite time.

Controller Design based on Generalized STA

Consider the following perturbed integrator system with relative degree $n-1$ with respect to output x_1

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{n-1} &= u + d
\end{aligned}
\tag{5.1}
$$

where $x_1, \cdots, x_{n-1}$ are the states of the perturbed integrator and d is the Lipschitz (in time) disturbance, which satisfied $|\dot{d}| < \Delta$.

Then n^{th} order Super-Twisting Control (n-STC) for the (5.1) is given as

$$
\begin{aligned}
u &= -k_1 \left|\phi_{n-2}\right|^{1/2} \operatorname{sign}\left(\phi_{n-2}\right) + x_n \\
\dot{x}_n &= -k_n \operatorname{sign}\left(\phi_{n-2}\right)
\end{aligned}
\tag{5.2}
$$

where ϕ_{n-2} is the same as (3.6).

Controller Design based on Generalized STA

After applying the control u and defining the new variable $z_n = x_n + d$ and taking the first time derivative of z_n, the system can be further written as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= -k_1 \left|\phi_{n-2}\right|^{1/2} \operatorname{sign}\left(\phi_{n-2}\right) + z_n \\ \dot{z}_n &= -k_n \operatorname{sign}\left(\phi_{n-2}\right) + \dot{d} \end{aligned} \tag{5.3}$$

The closed loop system (5.3) is the same as n-STA (3.6). Therefore, convergence condition remains the same for the proposed controller (5.2) as n-STA (3.6).

Simulation Results

For verifying the proposed technique of the n-STC following second and third order system are considered

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u_2 + d \end{aligned} \tag{5.4}$$

where x_1, x_2 are the states, u_2 is the control and $d = 2 + 3sin(t)$ is the Lipschitz (in time) disturbance. Similarly

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u_3 + d \end{aligned} \tag{5.5}$$

where x_1, x_2, x_3 are the states, u_3 is the control and $d = 2 + 3sin(t)$ is the Lipschitz (in time) disturbance.

Simulation Results

The controller for the systems (5.4) and (5.5) are designed as

$$u_2 = -k_1 \left|\phi_1\right|^{1/2} \operatorname{sign}(\phi_1) - \int_0^t k_3 \operatorname{sign}(\phi_1)\, d\tau \tag{5.6}$$

and

$$u_3 = -k_1 \left|\phi_2\right|^{1/2} \operatorname{sign}(\phi_1) - \int_0^t k_4 \operatorname{sign}(\phi_2)\, d\tau \tag{5.7}$$

where ϕ_1 and ϕ_2 are defined as (3.2) and (3.3) respectively.

Following parameters are used for the simulation

- uncertain double order integrator (5.4)
  - initial conditions $x_1(0) = -1$ and $x_2(0) = 3$
  - gains $k_1 = 6, k_2 = 4$ and $k_3 = 4$
- uncertain third order integrator (5.5)
  - initial conditions $x_1(0) = -1, x_2(0) = 3$ and $x_3(0) = 1$
  - gains $k_1 = 5, k_2 = 2, k_3 = 1$ and $k_4 = 4$

Simulation Results

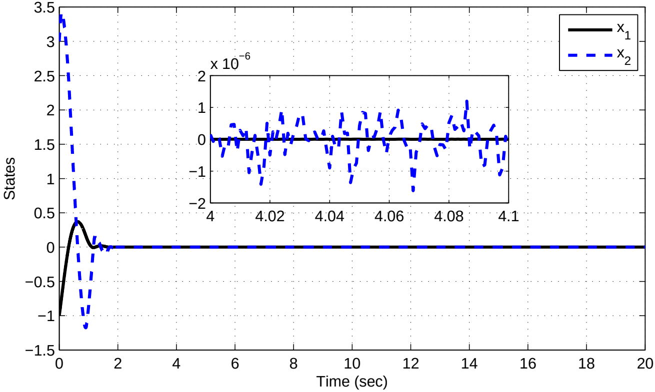


Figure : Evolution of states w.r.t., time (uncertain double integrator)

Simulation Results

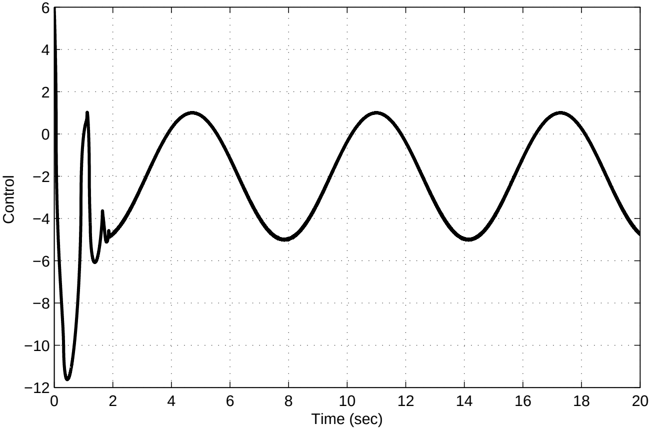


Figure : Evolution of control w.r.t., time (uncertain double integrator)

Simulation Results

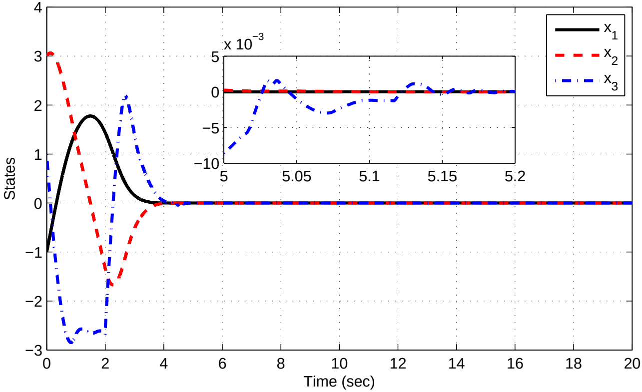


Figure : Evolution of states w.r.t., time (uncertain triple integrator)

Simulation Results

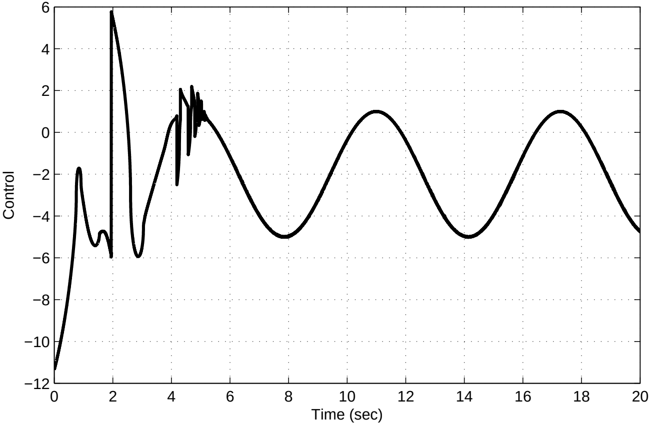


Figure : Evolution of control w.r.t., time (uncertain triple integrator)

Conclusion

- The paper discussed the realization of higher order sliding mode using the absolutely continuous control signal in the presence of matched Lipschitz (in time) uncertainties.
- For the above mentioned goal, generalization of the Super-Twisting algorithm (STA) for r relative degree system ensuring finite time convergence for the set $\sigma, \dot{\sigma}, ..., \sigma^{(r)}$ where σ represents the output using information of $\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}$ has been proposed.
- The convergence conditions for the 3-STA algorithm have been proposed.
- The formula for algorithm of arbitrary order has been also suggested.
- The Lyapunov function based convergence conditions for the 4 and higher STA are still open problem, which will look in the future.
- The simulations results are confirmed the efficiency of the proposed algorithm.

Thank You!